Comparison of Several Numerical Algorithms for Image Reconstruction from Irregular Samples

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Abstract:
Monte Carlo simulation is used to compare additive and multiplicative algebraic reconstruction technique (ART), and the Scatterometer Image Reconstruction (SIR) algorithms to reconstruct images from noisy irregular samples. SIR is the most robust in the presence of noise.

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It has been shown that additive algebraic reconstruction technique (AART) can completely recover an arbitrary band limited function for the noiseless case if the sampling is sufficiently dense: given adequate sampling, the reconstruction is essentially independent of the aperture function [2]. This result follows from the equivalence of AART and Gröchenig's irregular sampling reconstruction algorithm [2]. Following [2] we consider the relationship of several forms of the algebraic reconstruction technique (ART). Simulation is used to compare the performance of additive and multiplicative ART algorithms with the scatterometer image reconstruction (SIR) algorithm [6], a row-normalized derivative of multiplicative ART tailored to reduce the influence of noise on enhanced resolution image reconstruction.

Additive Versus Multiplicative ART

Gröchenig's reconstruction algorithm [3] is a general algorithm for reconstructing an image from irregular samples: assuming an appropriately dense sampling, Gröchenig's reconstruction algorithm states that the original bandlimited image can be perfectly reconstructed from irregularly distributed samples. It can be shown that block AART is equivalent to Gröchenig's algorithm [2]. AART, however, is only one example of algebraic reconstruction. Another ART algorithm is multiplicative ART (MART). Both algorithms attempt to reconstruct an image x from the observations or samples y. y consists of a (possibly irregular) sampling of the aperture filtered image which we can write as $y = Hx + \text{noise}$ where the operator H describes the aperture sampling. The image x is required to be band-limited. We assume the δ-dense sampling is sufficient for proper reconstruction of x from y according to Gröchenig's Lemma [2].

In this case, the essential difference between AART and MART is the regularization implicit in the algorithms. AART is equivalent to a least squares estimate in the limit of infinite iterations [1] based on the minimization problem:

$$\text{Minimize} \quad \|x^2\| \qquad\qquad \text{Subject to} \quad y = Hx.$$

MART with damping is a maximum entropy estimate in the limit of infinite iterations [1] based on the maximization problem:

$$\text{Maximize} \quad -\sum_{j=1}^{n} x_j \ln x_j \qquad \text{Subject to} \quad y = Hx.$$

In effect, AART makes no *a priori* assumptions about the data and fits the estimate strictly on the measurements available by minimizing the error of the back projection of the measurement onto the H space in the mean-squared error sense subject to $y = Hx$. Thus the reconstruction is strictly contained within the band-limited signal spectrum space defined by H.

On the other hand, MART effectively assumes a maximum entropy model for the data. In the frequency domain, the reconstruction is not strictly restricted to the band limited frequency domain spanned by the measurement space. Additional frequency content in the null space may be added by the algorithm to create a sharper image. However, the constraint $y = Hx$ remains, and the reconstruction is based on a projection of measurements onto the H space just as in AART.

The choice of least-squares versus maximum entropy over is debatable; however, we can, in principle, select any regularization to use in the reconstruction if the regularization fits with *a priori* knowledge. As discussed in [1], this decision may be based on the nature of the sampling mechanism and the nature of the solution the algorithm produces for under-determined systems. Thus, the selection is dependent on which regularization provides the best results for a given application. Least squares estimates produce a maximally smooth estimate where edges tend to be softened and blurred while a maximum entropy estimate produces a generally "sharper" image than least squares.

Since both forms of ART have the same constraint equation, the resulting solutions x are of the general form $x = U + Q$ where U is an element of the row space of H, or equivalently, the range space of the transpose of H, H', denoted $U \in \mathcal{R}(H')$; and Q is an element of the null space of H, denoted $Q \in \mathcal{N}(H)$. Any solution derived from either additive or multiplicative ART contains a component U. However, the solution derived by using AART results in $Q = 0$, while the solution from MART can have a non-zero Q component [5]. Since the constraint $y = Hx$ is the same for both algorithms, the solutions for both AART and MART are the same in the range space of H' in the limit of infinite iterations. The only difference between the AART and MART solutions is the Q component from the null space of H, which is defined by the nulls in the aperture function (assuming adequate sampling). If the aperture function does not have spectral nulls, the solutions are identical in the noise-free case. Thus both AART and MART can be used for reconstruction, with the understanding that in the null space, they may produce slightly different results based on the different regularizations.

Algorithm Performance Comparison

There are two key issues for reconstruction with iterative algorithms: finite number of iterations and noise. The former is a practical limitation since no iterative process can proceed indefinitely. While a particular algorithm converges to a particular solution in the limit, the limit may not be reached when the iteration is terminated. The result is an approximation to the optimal reconstruction, but may not be a complete reconstruction [1]. Truncation of the iterations is another form of regularization [5].

While Gröchenig's Lemma demonstrates that complete reconstruction of an irregularly sampled signal is possible, it does not consider the effects of noise. Although steps can be taken to minimize noise, noise limits the number of iterations that can be executed before the noise overtakes the reconstruction. Measurement noise changes the problem because noise is amplified along with the desired signal during the reconstruction. In effect, the reconstruction process acts as a high pass filter which removes the effects of the aperture and sampling functions. The high-pass nature of the reconstruction filter increases the noise power. In Wiener filtering, the reconstruction filter response is modified so that when a specified noise-to-signal ratio threshold is exceeded, the response is set to zero to minimize noise amplification [5].

Lacking a suitable theoretical analysis of the effects of noise, a Monte Carlo approach is employed to examine the behavior of the signal and noise power in the reconstruction. While a variety of other related reconstruction algorithms exist, we consider only one additional algorithm, the Scatterometer Image Reconstruction (SIR) algorithm. The algorithm is a derivative of MART developed for multivariate scatterometer image reconstruction with noisy measurements [6]. Although similar in performance to MART, SIR is more robust in the presence of noise, particularly at low signal to noise ratios, and is thus a useful alternative to AART and MART.

With noise, the performance of AART is significantly degraded, an observation originally motivated the development of SIR [6]. For both MART and SIR, the multiplicative update factors are damped which tends to reduce their sensitivity to noise. SIR further incorporates a non-linear damping which can further reduce the noise (at the expense of slower reconstruction).

Reconstruction Error

In general, iterative reconstruction suffers from two forms of error: reconstruction error and noise amplification. The reconstruction error is the difference between the iterative image estimate and the noiseless true image. Noise amplification results from the inverse filtering of the noise as previously noted [1, 5]. Based on our simulation results, at any given iteration, the reconstruction error is smaller and the noise amplification is greater for ARRT than for MART and SIR. The total error for AART reaches a minimum after just a few iterations, but grows rapidly as the iteration continues. SIR and MART reach minima in the total error more slowly but eventually achieve lower levels of total error. While the overall performance of the algorithms are similar, at lower reconstruction errors MART and SIR have lower noise amplification than AART. At the lowest reconstruction errors, SIR has the smallest noise. In all cases there is a tradeoff between reconstruction error and noise amplification controlled by the number of iterations. The differences become more apparent at lower signal to noise ratios.

It should be noted that while the rms error is an indicator of the accuracy of the reconstruction, the size and location of the error changes over the course of the iteration, depending on the regularization [5]. Also, the quality of the resulting imagery may not always be a direct function of total error [4]. The image quality for SIR at a given reconstruction error level is subjectively somewhat better than corresponding MART or AART products when used with scatterometer data [6].

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