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PHASE UNWRAPPING FOR MULTIDIMENSIONAL RATIONAL AND FINITE-LENGTH SEQUENCES

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ABSTRACT

Historically, the unwrapped phase for multidimensional sequences could only be computed by numerical integration of the phase derivative which often led to erroneous estimates of the unwrapped phase. In this paper a direct relationship between a multidimensional time series with finite-support and its unwrapped phase is shown. This relationship shows that the unwrapped phase of a multidimensional sequence is unique in the sense that once the phase at the origin is specified, the phase everywhere in the frequency domain follows. Additionally, the uniqueness of the unwrapped phase for multidimensional sequences which have a rational **Z** transform is shown. In either case, the unwrapped phase at a given point is shown to be computable using a real 1-d finite-length phase unwrapping proceedure based on Sturm sequence polynomials.

1. INTRODUCTION

Phase unwrapping is the determination of the continuous phase function of the complex **Z** transform of a signal sequence $x(n)$ [1]. Traditionally, the unwrapped phase of a multidimensional sequence could only be computed by numerical integration of the phase derivative [2]. In this paper a unique relationship between a *multi-dimensional* sequence with finite-support and its unwrapped phase is shown. A unique relationship between the polynomial coefficients of a multi-dimensional sequence with a rational **Z** transform and its unwrapped phase is also shown. Using the techniques presented in [4,5], the number of multiples of 2π which must be added to the principle value of the phase to obtain the unwrapped phase can be exactly computed in either case. The proofs provide methods for computing the unwrapped phase.

2. MULTIDIMENSIONAL SEQUENCES WITH FINITE SUPPORT

Consider the M-dimensional real sequence $x(n_1, ..., n_M)$ with a finite region of support, i.e., $x(n_1, ..., n_M) = 0$ outside the region $0 \le n_k < N_k$ for $k = 1, ..., M$. On the unit hypersphere, the M-dimensional **Z** transform $X(z_1, ..., z_M)$ of $x(n_1, ..., n_M)$ can be expressed as,

$$X(\omega_1, ..., \omega_M) = \sum_{n_1=0}^{N_1} \cdots \sum_{n_M=0}^{N_M} x(n_1, ..., n_M) e^{-j(n_1\omega_1 + \cdots + n_M\omega_M)} \tag{1}$$

where $z_k = e^{j\omega_k}$ for $k = 1, ..., M$.

Assuming that $|X(\omega_1, ..., \omega_M)| \neq 0$ on the unit hypersphere, the phase of $X(\omega_1, ..., \omega_M)$ relative to the phase at $\omega_1, ..., \omega_M = 0$ is,

$$\arg[X(\omega_1, ..., \omega_M)] - \arg[X(0, ..., 0)] = \arctan\left\{\frac{Im[X(\omega_1, ..., \omega_M)]}{Re[X(\omega_1, ..., \omega_M)]}\right\} + L(\omega_1, ..., \omega_M)\pi$$

The integer-valued function $L(\omega_1, ..., \omega_M)$ indicates the number of multiplies of π which must be added to the principle value of the phase of $X(\omega_1, ..., \omega_M)$ to produce a continuous phase function (the unwrapped phase) [1]. It will be shown that the function $L(\omega_1, ..., \omega_M)$, which defines the unwrapped phase, is unique, within an additive constant, for a given $x(n_1, ..., n_M)$. This result will be presented in the form of a theorem.

Theorem 1 Uniqueness of the Unwrapped Phase for a Sequence with Finite Support *Given an M-dimensional, real-valued sequence $x(n_1, ..., n_M)$ with finite support $0 \le n_k < N_k$, $k = 1, 2, ..., M$ that has a* **Z** *transform which is non-zero on the unit hypersphere, the unwrapped phase defined by $L(\omega_1, ..., \omega_M)$ is unique to within an additive multiple of 2π. Furthermore, when $x(n_1, ..., n_M)$ takes only rational values, $L(\omega_1, ..., \omega_M)$ can be exactly computed.*

Proof

Uniqueness in the one-dimensional case ($M = 1$) was first shown by McGowan and Kuc [3]. The details of the proof of uniqueness of $L(\omega)$ for a one-dimensional finite-length real sequence are presented in a companion paper [5] which also demonstrates that when $x(n)$ is rational-valued, $L(\omega)$ can be exactly computed. For multidimensional sequences when $M > 1$ it will be shown that the unwrapped phase at a given $\omega_1, ..., \omega_M$ is uniquely determined by a one-dimensional finite-length real phase unwrapping procedure.

Consider the point $s\pi = (s_1\pi, ..., s_M\pi)$ on the unit hypersphere with

$$\omega_i = s_i\pi, \ i = 1, 2, ..., M$$

where s_i is a rational number, $s_i = p_i/q_i$, $q_i > 0$, p_i and q_i integers. Note that the set of all such points $\{s\pi\}$ is dense on the unit hypersphere. Since the unwrapped phase is continuous, by showing that it is unique on $\{s\pi\}$ it follows that the unwrapped phase is unique everywhere on the unit hypersphere. A parametric line OS through the point $s\pi$ and the origin can be expressed,

$$\begin{aligned} s\omega &= (s_1\omega, ..., s_M\omega) \\ &= (\frac{p_1}{q_1}\omega, ..., \frac{p_M}{q_M}\omega). \end{aligned}$$

where ω is the line parameter. Along the line OS on the unit

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hypersphere, equation (1) can be written,

$$X(\omega_1,...,\omega_M)|_{OS} = \sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M} x(n_1,...,n_M)e^{-j(n_1 s_1+\cdots+n_M s_M)\omega}$$
$$= \sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M} x(n_1,...,n_M)e^{-j(n_1 p_1 Q_1+\cdots+n_M p_M Q_M)\omega/Q}$$
$$= \sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M} x(n_1,...,n_M)e^{-j(n_1 t_1+\cdots+n_M t_M)\omega'} \quad (2)$$

where

$$Q = \prod_{k=0}^{M} q_k, \quad Q_i = \prod_{\substack{k=0\\k\neq i}}^{M} q_k, \quad t_i = p_i Q_i$$

are integers and $\omega' = \omega/Q$. Any integer factors common to all of the t_i's and to Q can be removed from the t_i's and Q without affecting the result.

When one (or more) of the s_i's are less than zero, say $s_i < 0$ for $i = i_1, ..., i_2$, then equation (2) can be written as,

$$X(\omega_1,...,\omega_M)|_{OS} = e^{-j(N_{i_1}|t_{i_1}|+\cdots+N_{i_2}|t_{i_2}|)\omega'} \cdot \quad (3)$$
$$\sum_{n_1=0}^{N_1}\cdots\sum_{n_{i_1}=0}^{N_{i_1}}\cdots\sum_{n_{i_2}=0}^{N_{i_2}}\cdots\sum_{n_M=0}^{N_M} x(n_1,...,N_{i_1}-n_{i_1},...,$$
$$N_{i_2}-n_{i_2},...,n_M)e^{-j(n_1 t_1+\cdots+n_{i_1}|t_{i_1}|+\cdots+n_{i_2}|t_{i_2}|+\cdots+n_M t_M)\omega'}.$$

Note that equation (3) has the general form,

$$X(\omega_1,...,\omega_M)|_{OS} = e^{-jK_s\omega'}Y(\omega') \quad (4)$$

where

$$Y(\omega') = \sum_{k=0}^{K} y(k)e^{-jk\omega'},$$

$K = N_1|t_1| + \cdots + N_M|t_M|$, $K_s = N_{i_1}|t_{i_1}| + \cdots + N_{i_2}|t_{i_2}|$, and $y(k)$ is the one-dimensional, finite-length, real sequence,

$$y(k) = \sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M} y'(k, n_1, ..., n_M)$$

where

$$y'(k,n_1,...,n_M) = \begin{cases} x(n_1,...,N_{i_1}-n_{i_1},..., \\ \quad N_{i_2}-n_{i_2},...,n_M) & k = k_t \\ 0 & \text{otherwise} \end{cases}$$

with $k_t = (n_1|t_1| + \cdots + n_M|t_M|)$.

The multiplicative $e^{-jK_s\omega'}$ term in equation (4) produces an additive linear phase term in the unwrapped phase. Hence, the unwrapped phase of $X(\omega_1,...,\omega_M)$ along the line OS is the unwrapped phase of $Y(\omega')$ plus the linear phase term $K_s\omega'$, i.e.,

$$\arg[X(\omega_1,...,\omega_M)]|_{OS} - \arg[X(0,...,0)] = \arctan\left\{\frac{Im[Y(\omega')]}{Re[Y(\omega')]}\right\} + L(\omega')\pi - K_s\omega'$$

where $L(\omega')$ defines the unique unwrapped phase for the one-dimensional finite-length sequence $y(k)$. Computation of the unwrapped phase of $y(k)$ plus the linear phase term $K_s\omega'$ at the value of ω' corresponding to the point $s\pi$, gives the value of the unwrapped phase of $x(n_1,...,n_M)$ at the point $s\pi$.

Thus, determining the unwrapped phase of the multidimensional sequence $x(n_1,...,n_M)$ at the point $s\pi$ on the unit hypersphere, is equivalent to determining the unwrapped phase along the line OS at ω' of the one-dimensional finite-length sequence $y(k)$. Since the unwrapped phase for the one-dimensional finite length sequence $y(k)$ is unique, it follows that the unwrapped phase of $x(n_1,...,n_M)$ along the line OS and at the point $s\pi$ is unique. Furthermore, since the unwrapped phase in continuous and the set of points $\{s\pi\}$ is dense on the unit hypersphere, it follows that the unwrapped phase is unique everywhere on the unit hypersphere.

Note that when $x(n_1,...,n_M)$ is rational-valued, $y(k)$ will also be rational-valued. Hence, the techniques of [5] for exactly computing the unwrapped phase of a real, rational-valued, finite-length, one-dimensional sequence can be applied to exactly compute $L(\omega_1,...,\omega_M)$ for any point of the set $\{s\pi\}$.

3. RATIONAL Z TRANSFORMS

Let us now examine the computation of the unwrapped phase of a infinite-length sequence with a rational **Z** transform. Consider a general M-dimensional sequence $x(n_1,...,n_M)$ with a rational **Z** transform,

$$H(z_1,...,z_M) = \frac{N(z_1,...,z_M)}{D(z_1,...,z_M)} \quad (5)$$

where the numerator $N(z_1,...,z_M)$ and denominator $D(z_1,...,z_M)$ are of finite order,

$$N(z_1,...,z_M) = \sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M} a(n_1,...,n_M)z_1^{-n_1}\cdots z_M^{-n_M}$$
$$D(z_1,...,z_M) = \sum_{n_1=0}^{D_1}\cdots\sum_{n_M=0}^{D_M} b(n_1,...,n_M)z_1^{-n_1}\cdots z_M^{-n_M}$$

where $a(n_1,...,n_M)$ and $b(n_1,...,n_M)$ are real. It will be shown that the unwrapped phase of $x(n_1,...,n_M)$ is uniquely specified by the a and b coefficients. Note that in the previous section, I started with the values of the time domain sequence $x(n_1,...,n_M)$. For the case of a rational **Z** transform I begin with the values of the coefficients of the numerator and denominator of the **Z** transform, $a(n_1,...,n_M)$ and $b(n_1,...,n_M)$. $N(z_1,...,z_M)$ and $D(z_1,...,z_M)$ must be non-zero everywhere on the unit hypersphere.

Theorem 2 Uniqueness of the Unwrapped Phase for a Rational Z Transform *Given an M-dimensional, rational **Z** transform of the form given in equation (5) with real-valued coefficients $a(n_1,...,n_M)$ and $b(n_1,...,n_M)$ that is analytic on the unit hypersphere with both the numerator and denominator nonzero on the unit hypersphere, the unwrapped phase defined by $L(\omega_1,...,\omega_M)$ is unique within an additive multiple of 2π. Furthermore, when $a(n_1,...,n_M)$ and $b(n_1,...,n_M)$ are rational-valued, $L(\omega_1,...,\omega_M)$ can be exactly computed.*

Proof

Since $H(z_1,...,z_M)$ is analytic on the unit hypersphere, it has a continuous phase function and can be written,

$$H(z_1,...,z_M) = \frac{W(z_1,...,z_M)}{|D(z_1,...,z_M)|^2}$$

where

$$W(z_1,...,z_M) = N(z_1,...,z_M)D^*(z_1,...,z_M)$$

and where * indicate complex conjugation. On the unit hypersphere, the real and imaginary parts of $H(\omega_1,...,\omega_M)$ are,

$$\begin{aligned} Re[H(\omega_1,...,\omega_M)] &= \frac{1}{|D(\omega_1,...,\omega_M)|^2} Re[W(\omega_1,...,\omega_M)] \\ Im[H(\omega_1,...,\omega_M)] &= \frac{1}{|D(\omega_1,...,\omega_M)|^2} Im[W(\omega_1,...,\omega_M)]. \end{aligned}$$

Note that the phase of $H(\omega_1,...,\omega_M)$ is a function of the *ratio* of the real and imaginary parts of $H(\omega_1,...,\omega_M)$; hence, only the the ratio of the real and imaginary parts of $W(\omega_1,...,\omega_M)$ are of concern.

Note that $W(\omega_1,...,\omega_M)$ can be expressed as,

$$\begin{aligned} &W(\omega_1,...,\omega_M) \\ &= \Big[\sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M} a(n_1,...,n_M)e^{-j(n_1\omega_1+\cdots n_M\omega_M)}\Big]\cdot \\ &\quad \Big[\sum_{n_1=0}^{D_1}\cdots\sum_{n_M=0}^{D_M} b(n_1,...,n_M)e^{j(n_1\omega_1+\cdots n_M\omega_M)}\Big] \\ &= \sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M}\sum_{m_1=0}^{D_1}\cdots\sum_{m_M=0}^{D_M} a(n_1,...,n_M)b(m_1,...,m_M)\cdot \\ &\quad e^{j[(m_1-n_1)\omega_1+\cdots(m_M-n_M)\omega_M])} \\ &= e^{-j(N_1\omega_1+\cdots+N_M\omega_M)}\sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M}\sum_{m_1=0}^{D_1}\cdots\sum_{m_M=0}^{D_M}\cdot \\ &\quad a(N_1-n_1,...,N_M-n_M)b(m_1,...,m_M)\cdot \\ &\quad e^{j[(m_1+n_1)\omega_1+\cdots(m_m+n_M)\omega_M])} \end{aligned} \tag{6}$$

In the one-dimensional case, $M = 1$ and equation (6) becomes,

$$W(\omega) = e^{-jN\omega}\sum_{n=0}^{N}\sum_{m=0}^{D} a(N-n)b(m)e^{j(m+n)\omega} \tag{7}$$

which is of the form,

$$W(\omega) = e^{-jN\omega}Y(\omega)$$

where

$$Y(\omega) = \sum_{k=0}^{K} y(k)e^{jk\omega},$$

and $K = N + D$ where $y(k)$ is a one-dimensional, finite-length sequence,

$$y(k) = \sum_{i=\max(0,k-D)}^{\min(N,k)} a(N-n)b(k-i).$$

Hence, computation of the unwrapped phase of a 1-dimensional rational **Z** transform $H(\omega)$ is equivalent to computation of the unwrapped phase of the finite-length, real-valued sequence $y(k)$ where $y(k)$ is a simple function coefficients of the rational polynomial **Z** transform of $x(n)$. The additional exponential term $e^{-jN\omega}$ in equation (7) adds a linear phase term to the unwrapped phase of $y(k)$. The unwrapped phase of $y(k)$ is computed using the procedure described in [5]. The phase of $H(\omega)$ relative to the phase of $H(0)$ is,

$$\arg[H(\omega)] - \arg[H(0)] = \arctan\left\{\frac{Re[Y(\omega)]}{Im[Y(\omega)]}\right\} + L(\omega)\pi - N\omega$$

where $L(\omega)$ defines the unwrapped phase of $y(k)$. It follows that the unwrapped phase corresponding to $x(n)$ is unique to within an additive multiple of π. Note that when $a(n)$ and $b(n)$ are rational-valued, $y(k)$ will also be rational-valued. Hence, the techniques of [5] can be applied to exactly compute $L(\omega)$.

To determine the unwrapped phase of the rational **Z** transform for the case when $M > 1$, define the point $s\pi$ and the line OS as in Theorem 1. $W(\omega_1,...,\omega_M)$ along the line OS is then,

$$\begin{aligned} &W(\omega_1,...,\omega_M)\big|_{OS} = \\ &\sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M}\sum_{m_1=0}^{D_1}\cdots\sum_{m_M=0}^{D_M}\cdot \\ &\quad a(n_1,...,n_M)b(m_1,...,m_M)\cdot \\ &\quad e^{j[(m_1-n_1)t_1+\cdots(m_m-n_M)t_M]\omega'} \\ &= e^{-j(N_1+N_2+\cdots+N_M)\omega'}\sum_{n_1=0}^{N_1}\cdots\sum_{n_M=0}^{N_M}\sum_{m_1=0}^{D_1}\cdots\sum_{m_M=0}^{D_M}\cdot \\ &\quad a(N_1-n_1,...,N_M-n_M)b(m_1,...,m_M)\cdot \\ &\quad e^{j[(m_1+n_1)t_1+\cdots(m_m+n_M)t_M]\omega'} \end{aligned} \tag{8}$$

which, again, is of the form,

$$W(\omega_1,...,\omega_M)\big|_{OS} = e^{-jK_s\omega'}Y(\omega')$$

with

$$\begin{aligned} K_s &= N_1 + N_2 + \cdots + N_M, \\ K &= (N_1+D_1)t_1 + \cdots + (N_M+D_M)t_M, \end{aligned}$$

and

$$Y(\omega') = \sum_{k=0}^{K} y(k)e^{jk\omega'}$$

where $y(k)$ is a real-valued one-dimensional finite-length sequence uniquely defined by the a and b coefficients of the rational **Z** transform polynomial.

Hence, computing the unwrapped phase for the multidimensional rational **Z** transform is equivalent to computing the unwrapped phase along the line OS at the point $s\pi$ of the one-dimenional, finite-length, real sequence $y(k)$. The unwrapped phase of $H(\omega_1,...,\omega_M)$ where the point $(\omega_1,...,\omega_M)$ lies on the line defined by $s\omega'$, is,

$$\begin{aligned} &\arg[H(\omega_1,...,\omega_M)]\big|_{OS} - \arg[H(0,...,0)] = \\ &\arctan\left\{\frac{Re[Y(\omega')]}{Im[Y(\omega')]}\right\} + L(\omega') - K_s\omega' \end{aligned}$$

where $L(\omega')$ defines the unwrapped phase of the one-dimensional sequence $y(k)$. Since the phase of $y(k)$ is unique, it follows that the phase of $H(z_1,...,z_M)$ along the line OS and at the point $s\pi$ is unique. As before, since the unwrapped phase is continuous and the set $\{s\pi\}$ is dense on the unit hypersphere, it follows that the unwrapped phase of $H(\omega_1,...,\omega_M)$ is unique on the entire hypersphere.

Note that when $a(n_1,...,n_M)$ and $b(n_1,...,n_M)$ are rational-valued that $y(k)$ is rational-valued so that the modified Sturm sequence computations in [5] can be used to exactly compute $L(\omega_1,...,\omega_M)$ at any point of the set $\{s\pi\}$.

4. COMPUTATIONAL CONSIDERATIONS

Computation of the unwrapped phase at a given point of $\{s\pi\}$ requires: 1) computation of the coefficients of the Sturm sequence and 2) evaluation of the Sturm sequence at $\omega' = 0$ and the value of ω' corresponding to the desired point and 3) counting the number of sign changes in the Sturm sequence at each ω'. For many applications, the unwrapped phase at equally-spaced points on the unit hypersphere is desired. Repeating the process outlined above for each point can be computationally intensive. A significant computational savings can be realized by noting that many of the points of an equally-spaced lattice lie along the same lines through the origin (refer to Figure 1). For lattice points along these lines, the Sturm sequence need only be computed once. The Sturm sequence can be evaluated at $\omega' = 0$ and at each ω' corresponding to a lattice point on the line. The number of sign changes in the Sturm sequence between the origin and each point of the lattice along the line gives $L(\omega')$ at each of these points. In addition, the inherent symmetry in the phase function of a real sequence can also be used to minimize the number of points on the hypersphere at which the unwrapped phase needs to be computed from the Sturm sequence. For example, when $M = 2$ only the unwrapped phase in the first two quadrants need be computed. The unwrapped phase in the second two quadrants can be computed by symmetry considerations from the first two quadrants.

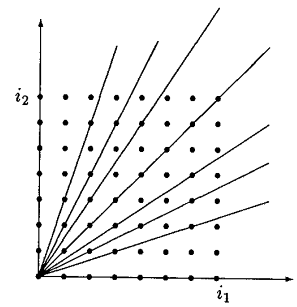

Figure 1: Lines passing through origin and (1,1), (1,2), (1,3), (2,1), (2,3), (3,1), and (3,2) pass through through several points of an 8×8 equally-spaced lattice in the first quadrant for $M = 2$. The points (i_1, i_2) and (i_1', i_2') lie along the same line if $i_1 i_2' = i_1' i_2$.

6. CONCLUSIONS

This paper has demonstrated that the unwrapped phase of a real multidimensional sequence with finite support is unique to within an additive multiple of 2π. In addition, the unwrapped phase for an infinite-support multidimensional sequence with a rational **Z** transform is shown to be a unique function of the **Z** transform coefficients. Using the numerical approach discussed in a companion paper [5], the integer-valued function $L(\omega_1, ..., \omega_M)$ which defines the unwrapped phase can be exactly computed when the sequence is rational-valued in the finite-support case or when the **Z** transform coefficients are rational-valued in the infinite-support case.

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